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LETTER TO THE EDITOR

On phase separation in the generalised spherical model

N Angelescu†, M Bundaru‡ and G Costache‡

† JINR Dubna, Moscow, Soviet Union

‡ Institute of Physics and Nuclear Engineering, Bucharest MG 6, Romania

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Abstract. It is proved that the interface in the generalised spherical model (the $D\to\infty$ limit of the D-vector model) with Kac–Helfand type interactions is diffuse at all temperatures. The interface width is shown to be proportional to the thickness of the sample. The magnetisation profile is also obtained.

A great deal of interest has been devoted recently to establishing the existence of a sharp interface and the roughening transition in different spin models (Gallavotti 1972, Dobrushin 1972, van Beijeren 1975, 1977, Weeks *et al* 1973, Abraham and Robert 1980, Abraham 1980). The existence of a sharp interface has been proved for the ferromagnetic Ising model in dimensions $d\geqslant 3$ (Dobrushin 1972, van Beijeren 1975). The same problem has been studied within the spherical model (introduced by Berlin and Kac (1952)) by Abraham and Robert (1980), who proved that the interface is in this case diffuse at all temperatures. They also showed that the magnetisation profile has some unphysical features, which led them to suggest that the model itself is inadequate for considering such 'non-translationally invariant' problems. Considering that the Berlin–Kac model can be obtained as the $D\to\infty$ limit of the D-vector isotropic model (the spherical limit) with homogeneous interactions (Kac and Thompson 1971), one may think that a more appropriate model for studying non-homogeneous problems can be obtained by looking at the spherical limit of the D-vector model with non-translationally invariant interactions. This limit has been recently considered (Knops 1973, Angelescu *et al* 1979), and the generalised spherical model (in which the overall constraint is replaced by mean local spherical constraints) has been obtained. This model is still exactly soluble in the sense that the state of the system can be determined once the solution of a nonlinear system for one-spin expectations is provided. However, the technical problem of handling this nonlinear system in the case of short-range interactions is extremely involved. Therefore we shall confine ourselves in this Letter to considering Kac–Helfand (1963) type interactions, which are technically easier to handle.

We shall prove that for the generalised spherical model with Kac–Helfand interactions the interface is diffuse at all temperatures. Having in mind the Ising model where Kac–Helfand interactions clearly favour the sharp interface, we can expect that the result will hold true also for the generalised spherical model with short-range interactions. To define the model, let us consider an array of $\mathcal{N}$ D-dimensional spins

$\{S_\rho\}_{\rho=1,\ldots,\mathcal{N}}$ of length $D^{1/2}$, whose interaction energy is given by

$$\mathcal{H}_D = -\tfrac{1}{2}\sum_{\rho,\rho'} \tilde{J}_{\rho\rho'} S_\rho S_{\rho'} - \sum_\rho \tilde{h}_\rho \sum_{\mu=1}^{D} S_\rho^{(\mu)}. \tag{1}$$

The spherical limit of the free energy per spin component equals the absolute minimum of

$$\mathcal{F}_{\mathcal{N}}(\tilde{\Gamma}) \equiv \frac{1}{\mathcal{N}}\left\{-\frac{1}{2\beta}\log\det\left(\frac{\beta}{2\pi}(\tilde{\Gamma}-\tilde{J})\right) + \tfrac{1}{2}(\tilde{h}, (\tilde{\Gamma}-\tilde{J})^{-1}\tilde{h}) + \tfrac{1}{2}\mathrm{Tr}(\tilde{\Gamma}-\tilde{J})\right\} \tag{2}$$

taken over all diagonal matrices $\tilde{\Gamma}$ for which $\tilde{\Gamma}-\tilde{J}>0$ (see however Angelescu *et al* (1979)). The function $\mathcal{F}_{\mathcal{N}}(\tilde{\Gamma})$ is strictly convex and has a unique minimum point $\tilde{\Gamma}^{(0)}$, given by the solution of the system

$$\beta^{-1}(\tilde{\Gamma}-\tilde{J})^{-1}_{\rho\rho} = 1 - [(\tilde{\Gamma}-\tilde{J})^{-1}\tilde{h}]^2_\rho. \tag{3}$$

One identifies $[(\tilde{\Gamma}^{(0)}-\tilde{J})^{-1}\tilde{h}]_\rho \equiv m_\rho$ with the local magnetisation at site ρ.

For our purpose we shall consider $\mathcal{N}=NM$ spins S_{ir} arranged in M parallel layers, each layer having N spins; i labels the layers and r the in-layer position of a spin. The interaction among spins is supposed to be

$$\tilde{J}_{ir,i'r'} = J_{ii'}P_{rr'},$$

$$J_{ii'} = \tau\delta_{ii'} + \delta_{|i-i'|,1}, \qquad P_{rr'} = 1/N. \tag{4}$$

We shall also consider the boundary layer spins as acted upon by oppositely directed magnetic fields, $h_i = H(\delta_{i1}-\delta_{iM})$, and we shall let $H\to\infty$ at the end of the calculations. The in-layer homogeneity leads to $\tilde{\Gamma}^{(0)}_{ir,ir} = \gamma_i^{(0)}(N)$ with $\{\gamma_i^{(0)}(N)\}$ the unique solution of the system

$$(\beta\gamma_i)^{-1} = 1 - [(\Gamma-J)^{-1}h]_i^2 - (\beta N)^{-1}[(\Gamma-J)^{-1}_{ii} - \gamma_i^{-1}], \qquad i=1,2,\ldots,M, \tag{5}$$

in the domain $\Gamma - J>0$; here $\Gamma_{ij}=\gamma_i\delta_{ij}$. Let us note that $\{\gamma_i^{(0)}(N)\}$ are symmetric and $m_i = [(\Gamma_N^{(0)}-J)^{-1}h]_i$ are antisymmetric with respect to the middle plane.

Next, we have to perform the thermodynamic limit ($N\to\infty$). In this respect it suffices to note that the minimum in (2) and the limit $N\to\infty$ can be taken in either order, whereupon the minimum point itself, $\{\gamma_i^{(0)}(N)\}$, converges to the unique minimum point of $\mathcal{F}(\Gamma) = \lim_{N\to\infty}\mathcal{F}_{NM}(\Gamma)$. Therefore, after performing the thermodynamic limit, one should study the absolute minimum of

$$\mathcal{F}(\Gamma) = \frac{1}{2M}\left[-\frac{1}{\beta}\log\det\left(\frac{\beta}{2\pi}\Gamma\right) + ((\Gamma-J)^{-1}h, h) + \mathrm{Tr}\,\Gamma\right] \tag{6}$$

on $\bar{\mathcal{D}} = \{\Gamma|\Gamma-J\geqslant 0, \Gamma_{ij}=\gamma_i\delta_{ij}\}$. Denoting by $\Gamma^{(0)}$ the minimum point of $\mathcal{F}(\Gamma)$, one can calculate the layer magnetisations as

$$m_i^{(M)} = [(\Gamma^{(0)}-J)^{-1}h]_i, \qquad i=1,2,\ldots,M;$$

we note that this expression makes sense even when $\Gamma^{(0)}-J$ is singular, since $\Gamma^{(0)}-J$ has a simple spectrum and is centro-symmetric, while h is centro-antisymmetric.

We can now state the main results as follows:

(i) $m^{(M)}_{[M/2]+i} = \mathrm{O}(1/M)$ for all β and all fixed i;

(ii) $\lim_{M\to\infty} m^{(M)}_{[Mx]} = m_{\mathrm{B}}(\beta)\cos\pi x$, where $m_{\mathrm{B}}(\beta)$ is the bulk magnetisation;

(iii) $\lim m_i^{(M)} = m_i(\beta)$ and $m_i(\beta)$ approaches exponentially $m_{\mathrm{B}}(\beta)$ when $i\to\infty$ and $\beta\neq\beta_{\mathrm{c}}$.

The result (i) implies that the interface is diffuse at all temperatures, (ii) shows that the interface width is of the order of M, while (iii) points out that the boundary conditions are felt effectively within a finite distance (proportional to the correlation length of the bulk) from the surfaces. Thus the non-physical behaviour obtained in the model studied by Abraham and Robert (1980) is no longer present.

We shall indicate below the idea of the proof. As maybe already noticed, one of the main steps consists in finding the minimum point $\Gamma^{(0)}$ of $\mathscr{F}$, which can be either in the interior of $\bar{\mathscr{D}}$ or on its frontier. Since there is a certain disparity between these cases, they seemingly cannot be handled on an equal footing. However this is not entirely true; provided we study closely the $N\to\infty$ limit of $\Gamma_N^{(0)}$ (which is $\Gamma^{(0)}$), we can hope to obtain a unique description of both situations mentioned. To this end, note that

$$\lim_{N\to\infty} N^{-1}[(\Gamma_N^{(0)}-J)^{-1}]_{ii}>0 \qquad \text{provided } \lim_{N\to\infty}(N\lambda_{\min}^{(N)})^{-1}>0,$$

where $\lambda_{\min}^{(N)}$ is the minimum eigenvalue of $\Gamma_N^{(0)}-J$; in this case $\alpha_i\equiv\lim_{N\to\infty}[N^{-1}(\Gamma_N^{(0)}-J)_{ii}^{-1}]^{1/2}$ will be the eigenvector of $\Gamma^{(0)}-J$ corresponding to the eigenvalue equal to zero. Taking account of (5), it may be seen that if $\Gamma^{(0)}$ is on the boundary of $\bar{\mathscr{D}}$ it satisfies the system

$$(\beta\gamma_i)^{-1}=1-m_i^2-\alpha_i^2, \qquad i=1,2,\ldots,M, \tag{7}$$

where m_i and $\alpha_i>0$ are functions of Γ satisfying

$$(\Gamma-J)m=h, \tag{8}$$

$$(\Gamma-J)\alpha=0. \tag{9}$$

On the other hand, when $\lim_{N\to\infty}(N\lambda_{\min}^{(N)})^{-1}=0$ one has $\Gamma^{(0)}-J>0$ and (7)–(9) still hold true (in this case by (9), $\alpha_i=0$). Thus the minimum point problem reduces to solving the system (7)–(9). Observing that equations (8) and (9) are in fact recurrence relations (see equation (4)), the system (7)–(9) can be brought into the form

$$\begin{aligned} &\boldsymbol{r}_{i+1}=(\boldsymbol{r}_i/r_i)F_\beta(r_i)-\boldsymbol{r}_{i-1}, \qquad i=1,2,\ldots,M,\\ &\boldsymbol{r}_0=(H,0)\boldsymbol{r}_{M+1}=(-H,0),\\ &\boldsymbol{r}_i=(m_i,\alpha_i), \qquad i=1,2,\ldots,M,\end{aligned} \tag{10}$$

and

$$F_\beta(x)=x/\beta(1-x^2)-\tau x. \tag{11}$$

Let us consider further the function $\phi_\beta:R^2\times S\to S\times R^2$ where S is the open unit disc of R^2, defined by

$$\phi_\beta(\boldsymbol{r},\boldsymbol{r}')=(\boldsymbol{r}',(\boldsymbol{r}'/r')F_\beta(r')-\boldsymbol{r}), \tag{12}$$

which is a diffeomorphism. Its fixed points are

(a) $(0,0)$ for $\beta\le\beta_c\equiv(\tau+2)^{-1}$,

(b) $(0,0)$ and the set $\{(\boldsymbol{r},\boldsymbol{r})\,|\,r=m_B(\beta)\}$ for $\beta>\beta_c$.

It can be shown that when $H\to\infty$ the system (10) becomes

$$\begin{aligned}&\phi_\beta(\boldsymbol{r}_{i-1},\boldsymbol{r}_i)=(\boldsymbol{r}_i,\boldsymbol{r}_{i+1}), \qquad i=2,3,\ldots,M-1,\\ &\boldsymbol{r}_1=(1,0), \qquad \boldsymbol{r}_M=(-1,0).\end{aligned} \tag{13}$$

We are therefore interested in finding the properties of long 'trajectories' of ϕ_β, i.e. sequences $\{\boldsymbol{r}_i\}_{i=1,\ldots,M}$ with arbitrary large M. The proof rests heavily on showing that these trajectories must 'spend much time' near the fixed points. In order to do this one can use the fact that the trajectories of ϕ_β are convex outside the circle of fixed points and concave inside. This implies easily that $\alpha = 0$ for $\beta < \beta_c$ and M large enough, in which case the system (10) reduces to the layer magnetisations system studied in much detail by Angelescu *et al* (1981). It is somewhat more complicated to show that for $\beta > \beta_c$ the long trajectories look typically as in figure 1; the trajectory points are practically uniformly distributed over angles and the outer points approach exponentially the circle of radius $m_B(\beta)$. This is exactly what we need in order to prove (i)–(iii).

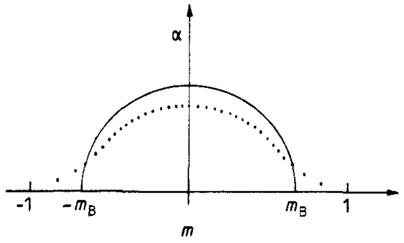

Figure 1. A typical trajectory of ϕ_β. $M = 37$.

Let us finally remark that the D-vector model with Kac–Helfand interactions is suited for a similar analysis, with α_i playing the role of perpendicular magnetisations with respect to the magnetic field direction; one can assert that for every $D \geqslant 2$ the interface is also diffuse. Loosely speaking, the local order parameters have the possibility to build up a Bloch wall of width M with low energy cost.

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